Chapter 7

CHARACTERIZING DYNAMIC IRRIGATION POLICIES VIA GREEN'S THEOREM

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Abstract We derive irrigation management schemes accounting for the dynamic response of biomass yield to salinity and soil moisture as well as for the cost of irrigation water. The simple turnpike structure of the optimal policy is characterized using Green's Theorem. The analysis applies to systems of arbitrary end conditions. A numerical application of the turnpike solution to sunflower growth under arid conditions reveals that by selecting the proper mix of fresh and saline water for irrigation, significant savings on the use of freshwater can be achieved with negligible loss of income.

1. Introduction

Increasing water scarcity and the alarming deterioration in the quality of many freshwater resources call for improved irrigation efficiency to sustain viable agriculture over vast areas around the globe. Using water of lesser quality and continuously adjusting irrigation rates to the varying needs of the growing

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plants can save a significant fraction of freshwater used in traditional irrigation practices. The trade-offs between the cost of water and the essential contribution of suitable soil moisture to biomass growth give rise to optimization problems that are both theoretically interesting and practically relevant.

An Optimal Control analysis of a dynamic irrigation problem accounting for soil moisture and biomass growth dynamics as well as for the associated cost of irrigation water is presented in Shani, Tsur and Zemel (2004), where the ensuing optimal policy is shown to take a particularly simple form. The policy is defined in terms of two parameters: a turnpike soil moisture $\hat{\theta}$ and a stopping date, such that the optimal moisture process, $\theta(t)$, must be brought from its initial level to the turnpike $\hat{\theta}$ *as rapidly as possible* and maintained at that level until the stopping date, at which time irrigation ceases and the plants are left to grow on the remaining soil moisture until the time of harvest. This simple turnpike behavior is neither unique to the irrigation problem nor is it rare in the dynamic optimization literature. Similar characterizations have been derived for a large variety of economic and management problems (see e.g. Vidale and Wolfe (1957), Sethi (1974), Haruvy, Prasad and Sethi (2003)) and explained by geometrical considerations using Green's Theorem (see Miele (1962), Sethi (1973), Sethi (1977), Sethi and Thompson (2000)). In this method, one eliminates the control, replaces the line integral in the objective functional by an area integral, and compares the values obtained from any two feasible policies by analyzing the sign of the integrand over the area encircled by the trajectories corresponding to these policies. The similar characteristics of the optimal irrigation policy suggest that this problem can also be analyzed in terms of Green's theorem. It turns out that certain features of the irrigation problem render the application of Green's Theorem non trivial in this case, and novel considerations must extend the method to derive the optimal policy.

The purpose of the present work is twofold. First, we adapt the standard Green's Theorem analysis to more complex situations of arbitrary end conditions. Applying the method to the irrigation problem, we explain the simple structure of the optimal policy and provide new links relating the method and Optimal Control theory. Second, we extend the original model of Shani, Tsur and Zemel (2004) by incorporating salinity effects. In many arid regions, brackish water is available to replace scarce freshwater resources for irrigation purposes. The growers, however, must also account for the reduction in yield implied by increased salinity in the root zone. We find that by carefully adjusting the salinity of the irrigation water mix and the parameters of the turnpike policy, the growers can increase the net income from their crop and, more importantly, mitigate freshwater scarcity (manifest in terms of exogenous freshwater quota imposed for each growing season) with only minor income loss.

2. The irrigation management problem

Let $m(t)$ represent the plant biomass at time $t \in [0,T]$, where T denotes the time from emergence to harvest. Marketable yield is derived from the biomass according to the increasing yield function $y(m)$ If yield and biomass are the same, then $y(m) = m$. Often, however, $y(m)$ vanishes for m below some threshold level, but above this level it increases at a rate that exceeds that of the biomass. At each point of time the biomass grows at a rate that depends on the current biomass state as well as on a host of factors including availability of water, salinity, sunlight intensity, day length and ambient temperature. Some of these factors (e.g. soil water content and salinity) can be controlled by the growers who derive irrigation water from two alternative sources: a costly supply of freshwater, and a cheaper supply of saline water from a local aquifer or a wastewater recycling plant. We assume that the mix of water from the two sources is determined at the beginning of the growing season and this fixes the salinity of irrigation water for the entire growing season.

Using the fraction s of saline water as a proxy for the salinity of the irrigation water mix, denoting by $\theta(t)$ the water content in the root zone, and taking all factors that are beyond the growers' control as given, the plant biomass rate of growth depends on $\theta(t)$ and $m(t)$ according to

$$\frac{dm(t)}{dt} \equiv \dot{m}(t) = q(s)g(\theta(t))h(m(t)) \tag{7.1}$$

Implicit in (7.1) is the assumption that the biomass growth rate can be factored to terms depending on s, θ and m separately. For fresh water, the function q is normalized at $q(0) = 1$. Since low salinity bears minor effects while excessive salinity hampers growth, we assume that q is decreasing, with $q'(0) = 0$. The functions g and h are assumed to be strictly concave in their respective arguments, with g vanishing at the wilting point $\theta_{\min}$ and obtaining a maximum at some value $\theta_{\max}$ (too much moisture harms growth). Thus, $g(\theta_{\min}) = 0$, $g'(\theta_{\max}) = 0$, $g'(\theta) > 0$ for $\theta \in (\theta_{\min}, \theta_{\max})$ and $g''(\theta) < 0$ for all θ. Since it is never optimal to increase moisture above the maximum level, we restrict attention to processes with $\theta \leq \theta_{\max}$.

The dynamics of water content in the root zone is determined by mass conservation, implying that the change in $\theta(t)$ at each point of time must equal water input through irrigation, $x(t)$, minus losses due to evapotranspiration, (ET) and drainage (D). (Rainfall can also be incorporated in this framework, but to focus on irrigation management we assume no rainfall.)

Evapotranspiration rate is specified as

$$ET(\theta, m) = \beta q(s)g(\theta)f(m) \tag{7.2}$$

where the coefficient β depends only on climatic conditions and is independent of s, m and θ and $0 \leq f(m) \leq 1$ is a crop scale factor representing the degree of leaves exposure to solar radiation (Hanks (1985)). The use of the same factor $q(s)g(\theta)$ in (7.1) and (7.2) is based on the linear relation between biomass production and evapotranspiration (see deWit (1958)).

The rate of water drainage $D(\theta)$ is assumed to be positive, increasing and convex for the relevant soil moisture range. When all the flow rates are measured in mm day^{-1}and θ is a dimensionless water concentration, the soil water balance can be specified as

$$Z\dot{\theta}(t) = x(t) - \beta q(s)g(\theta(t))f(m(t)) - D(\theta(t)) \tag{7.3}$$

where Z is the root depth and $Z\theta(t)$ measures the total amount of water in the root zone (mm).

Let W_f and W_s denote unit prices of fresh and saline water, respectively, assumed fixed throughout the growing season. For the chosen fraction s of saline water, the growers pay the price $W = (1-s)W_f + sW_s$. At harvest time T they also receive the revenue $py(m(T))$, where p is the output price. For a growing season that lasts a few months we can ignore discounting, and the return to water (excluding expenses on inputs other than water) is $py(m(T)) - W\int_0^T x(t)dt$.

Given the salinity s, we define the relative cost of water $w = W/p$ and formulate the irrigation management problem as finding the irrigation policy $\{x(t), 0 \leq t \leq T\}$ that maximizes

$$V(m_0, \theta_0) = Max_{\{x(t)\}} \left\{ \int_0^T -wx(t)dt + y(m(T)) \right\} \tag{7.4}$$

subject to (7.1), (7.3), $m(0) = m_0$, $\theta(0) = \theta_0$ and $0 \leq x(t) \leq \bar{x}$, where $m_0 > 0$ and $\theta_{\min} \leq \theta_0 \leq \theta_{\max}$ are the initial biomass and soil moisture levels and $\bar{x}$ is an upper bound on the feasible irrigation rate, reflecting physical constraints on irrigation equipment or on soil water absorption capacity. The upper bound $\bar{x}$ exceeds the water loss terms of (7.3) throughout the relevant ranges of m and θ so $\theta(t)$ increases when $x = \bar{x}$ (violations of equivalent assumptions in related contexts are discussed in Sethi (1977)). In the following section we characterize the optimal irrigation policy corresponding to (7.4), with one control variable (the irrigation rate x) and two state variables (the biomass m and the moisture θ). The optimal water mix and the corresponding salinity are considered in a later section.

3. Solution by Green's Theorem

An Optimal Control analysis of the optimization problem (7.4) is presented in Shani, Tsur and Zemel (2004), where the optimal turnpike policy is derived and explained in terms of the linear dependence of the objective and state equations on the control variable x, which gives rise to the typical Most Rapid Approach Path (Spence and Starrett (1975)). The similarity of this policy to the characteristic behavior derived by Sethi and coworkers for a large variety of optimization problems using Green's Theorem suggests that the irrigation problem can also be analyzed by means of this method. We note, however, that certain features of the irrigation problem require proceeding beyond the standard application of Green's Theorem in order to derive the optimal policy.

First, the problem involves two state variables m and θ hence the relevant (state-time) space is three-dimensional. In the standard approach, one would employ Stokes' Theorem instead of Green's, as in Sethi (1976). Here, however, we observe in (7.1) that the biomass process $m(t)$ evolves monotonically in time, hence m can serve as an effective time index, reducing the analysis to the two-dimensional (m,θ) state-space.

Second, the final values of the state variables are free in this problem. Thus, two arbitrary feasible trajectories need not end at the same point and joining them may not give rise to the closed trajectory that the method requires. Moreover, the determination of the turnpike moisture state is further complicated by its dependence on the transversality conditions corresponding to the free endpoints. Finally, fixing the endpoint of the trajectories in the (m,θ) space does not determine the time it takes the processes to get there. Comparing the values derived from trajectories of different durations does not provide the required information, and one must impose the correct duration T to obtain a feasible plan. As we shall see, this constraint introduces an additional term to the effective objective function, which turns out to be instrumental in the determination of the turnpike.

We proceed now to characterize the optimal irrigation policy using Green's Theorem while accounting for the new features listed above. Although the derivation is carried out for the specific irrigation problem presented above, we note that the considerations apply to a large class of optimization problems that are linear in the control or, more generally, satisfy the conditions of Corollary 2.1 of Sethi (1977) and have arbitrary end conditions (free or fixed duration, free or fixed final states, time or state dependent salvage functions etc.). In particular, we study the properties of the family of trajectories which consist of at most three distinct segments: (i) a "nearest approach" segment, leading from the initial state (m_0,θ_0) to some arbitrary turnpike moisture state $\hat{\theta}$ with $x(t)=0$ if $\theta_0>\hat{\theta}$ and $x(t)=\bar{x}$ if $\theta_0<\hat{\theta}$; (ii) a singular segment, which maintains the moisture process fixed at the turnpike state $\hat{\theta}$ by setting

$x(t) = \beta q(s)g(\hat{\theta})f(m(t)) + D(\hat{\theta})$; and (iii) a "nearest exit" segment leading from the turnpike to some final state with $x = 0$ or $x = \bar{x}$. This family, which includes also trajectories in which one or two segments are skipped (e.g. when the turnpike state coincides with the initial or final moisture levels), is termed the family of *turnpike processes*. It turns out that turnpike processes include the optimal policy as well as the processes of extreme duration leading to any given final state. Since the analysis involves processes of arbitrary duration, we shall refer to trajectories consistent with all the constraints of problem (7.4) except for the duration T, as *dynamically feasible*. Evidently, all turnpike processes are dynamically feasible.

We omit, for brevity, the salinity argument from q and the time index from all functions. We use (7.1) and (7.3) to eliminate the control and write the value obtained from any dynamically feasible trajectory Γ initiated at (m_0, θ_0) and ending at some arbitrary final state (m_F, θ_F) with $m_F \geq m_0$ and $\theta_F \leq \theta_{\max}$ as:

$$V_\Gamma = \int_\Gamma -\{[\frac{wD(\theta)}{qg(\theta)h(m)} + \frac{w\beta f(m)}{h(m)}]dm + wZd\theta\} + y(m_F) \qquad (7.5)$$

(cf. Hermes and Haynes (1963)). It follows that the difference between the values obtained from any two dynamically feasible trajectories with the same initial and final states (but not necessarily of the same duration) can be evaluated, using Green's Theorem, by

$$\Delta V = \int\int_\sigma \frac{g'(\theta)}{qh(m)g^2(\theta)} w\xi(\theta)d\sigma \qquad (7.6)$$

where σ is the area encircled by the graphs of the two trajectories and

$$\xi(\theta) = g(\theta)D'(\theta)/g'(\theta) - D(\theta) \qquad (7.7)$$

is an increasing function.

Searching for the roots of the integrand of (7.6) will not yield the correct turnpike state because the trajectories may be, as noted above, of different durations. We can, however, follow the same procedure to obtain the time difference. Recalling (7.1), we write

$$T = \int_0^T dt = \int_\Gamma \frac{dm}{qg(\theta)h(m)} \qquad (7.8)$$

Using Green's Theorem again, we find

$$\Delta T = \int\int_\sigma \frac{g'(\theta)}{qh(m)g^2(\theta)} d\sigma \qquad (7.9)$$

Hence, for any given constant H

$$\Delta V - H\Delta T = \int\int_{\sigma} \frac{g'(\theta)}{qh(m)g^2(\theta)}[w\xi(\theta) - H]d\sigma \tag{7.10}$$

Since the integrand of (7.9) is positive for all $\theta < \theta_{\max}$ it follows that the minimal duration of a dynamically feasible process leading to (m_F, θ_F) must correspond to a turnpike process. To see this, extend the increasing nearest approach segment all the way to $\theta_{\max}$. Consider now the decreasing nearest exit segment ending at (m_F, θ_F). If the two segments cross below $\theta_{\max}$ the trajectory comprising these segments is the dynamically feasible process of shortest duration connecting the initial and final states, since any other dynamically feasible process with the same endpoints must lie below its graph. If the two segments do not cross below $\theta_{\max}$ the minimum duration is obtained by the three-segment turnpike process comprising them and the singular segment connecting them at $\theta_{\max}$.

Similar considerations involving the decreasing nearest approach segment and the increasing nearest exit segment ending at (m_F, θ_F), imply that the maximum dynamically feasible duration is also obtained by a turnpike process. Moreover, the durations of turnpike processes ending at (m_F, θ_F) vary continuously with the turnpike level of their corresponding singular segments. We have, therefore established

PROPOSITION 7.1 *For any dynamically feasible process ending at (m_F, θ_F) there exists a corresponding turnpike process of the same duration and the same final state.*

The values obtained from the two processes of Proposition 7.1 can now be compared:

PROPOSITION 7.2 *The corresponding turnpike process of any dynamically feasible process yields higher (or equal, if the latter process is itself a turnpike process) value than the original process.*

PROOF: Choose the constant H of (7.10) as $H = w\xi(\hat{\theta})$, where $\hat{\theta}$ is the turnpike state associated with the singular segment of the turnpike process. With $\Delta T = 0$, (7.10) reduces to $\Delta V = \int\int_{\sigma} \frac{g'(\theta)}{qh(m)g^2(\theta)}[w\xi(\theta) - H]d\sigma$. Recalling that ξ is increasing, we see that the integrand is positive for all states above the singular segment and negative below it (Figure 7.1). With the counterclockwise convention for closed line integrals, we see that for each of the closed sub-areas encircled by the two trajectories, the turnpike process yields a larger value. Thus, the result follows for the entire trajectories as well. □

The characterization of the optimal policy follows immediately from Proposition 7.2:

PROPOSITION 7.3 *The optimal policy must be a turnpike process.*

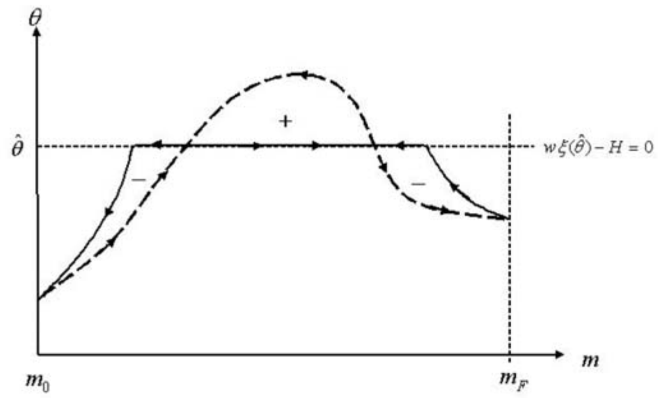

Figure 7.1. Comparing the values obtained from an arbitrary dynamically feasible process (dashed line) and the corresponding turnpike process (solid line). The "+" and "−" symbols indicate the sign of the integrand, and the arrows indicate the direction of the line integration in each of the closed sub-areas. The constant H is adjusted so that the integrand vanishes on the singular segment.

The role of the constant H ought to be explained. In Shani, Tsur and Zemel (2004) we show that the value of H corresponding to the optimal policy equals the constant value of the Hamiltonian under this policy. From Optimal Control theory we know that the Hamiltonian can be regarded as the shadow price associated with a marginal increase in T. In the context of (7.10), this shadow price assumes the role of the Lagrange multiplier associated with the constraint that the duration must be fixed at the given time T. Indeed, with $\Delta T = 0$ the term including H in the right-hand-side of (7.10) gives a vanishing contribution for any value of this constant (see (7.9)). Nevertheless, the particular choice of the Hamiltonian value for H ensures that the integrand has a definite sign for each sub-area, as required by Green's Theorem method.

While Proposition 7.3 characterizes the optimal policy as a turnpike process, it does not provide specific information on the nearest exit segment. In Shani, Tsur and Zemel (2004) we establish that this segment must be decreasing, so it is optimal to cease irrigation prior to T and reduce the moisture level towards the end of the growing season. This is indeed a common practice among growers. The property is most easily demonstrated via the transversality condition associated with the free final moisture state; it will not be further considered here.

Unlike previous work based on Green's Theorem, the turnpike state is not explicitly specified in this problem, because neither the final state nor the Hamiltonian are a priori given. In fact, the full dynamics of the state and costate variables, as well as the relevant transversality conditions must be invoked to derive the turnpike and the final states. Nevertheless, the power of the method to derive and explain the simple structure of the optimal policy is evident also for the more complicated problem considered here.

4. Salinity and scarcity

We turn now to study the effects of salinity on the optimal policy. Since $q'(0) = 0$, mixing a small amount of saline water must have a negligible effect on plant growth, yet it helps to reduce the cost of irrigation water. Higher salinity levels hamper growth and reduce yields. These trade-offs suggest an internal solution for the optimal salinity. We illustrate these trade-offs by applying the model to the growth of Ornamental sunflower (*Helianthus annuus* var dwarf yellow) in the Arava Valley in Israel. Lack of precipitation throughout the growing period and deep groundwater (120 *m* below soil surface) ensure that irrigation is the only source of water. The biomass and moisture dynamics are modeled using the functional specifications of Shani, Tsur and Zemel (2004):

$$\dot{m} = q(s)(1.21\Theta - 1.71\Theta^2)m(1 - m/491) \quad (7.11)$$

and

$$\dot{\theta} = [x - 0.19q(s)(1.21\Theta - 1.71\Theta^2)m(1 - m/785.6) - 3600\Theta_d^{5.73}]/600 \quad (7.12)$$

where $\Theta = (\theta - 0.09)/0.31, \Theta_d = (\theta - 0.04)/0.36$ and $q(s) = 1/\left[1 + (4s/3)^3\right]$ (Dudley and Shani (2003)).

Marketable yield for sunflowers is obtained only at biomass levels above 350 $g \cdot m^{-2}$. At the maximal biomass ($m = 491\ g \cdot m^{-2}$) the yield comprises 80% of the biomass. Assuming a linear increase gives rise to the following yield function:

$$y(m) = \begin{cases} 0 & \text{if } m < 350\ g \cdot m^{-2} \\ 2.79(m - 350) & \text{if } m \geq 350\ g \cdot m^{-2} \end{cases} \quad (7.13)$$

The initial soil water and biomass levels were taken at $\theta_0 = 0.1$ (just above water content at the wilting point $\theta_{\min} = 0.09$, where Θ and the growth rate vanish) and $m_0 = 10\ g \cdot m^{-2}$ (about 2% of the maximal obtainable biomass). The maximal feasible irrigation rate is $\bar{x} = 41.8\ mm \cdot day^{-1}$ and the growing period lasts 45 days. Sunflower seeds are sold at about \$1 kg^{-1}, yielding the relative water prices of $w_f = W_f/p = 0.3\ kg \cdot m^{-3}$ (freshwater) and $w_s = W_s/p = 0.1\ kg \cdot m^{-3}$ (saline water).

Using fresh water only, a numerical implementation of the optimal policy based on the above specifications gave rise to the turnpike level $\hat{\theta} = 0.148$. Irrigating at the maximal rate brings soil moisture to the turnpike at $t_1 = 0.7$ day, at which time irrigation rate is tuned so as to maintain the soil water content fixed at $\hat{\theta}$ for the major part of the growing period of $T = 45$ days. However, during the last 2.8 days irrigation is avoided because the gain in yield due to continued irrigation is not sufficient to cover the cost of the water needed to maintain the high soil water content. The corresponding harvested yield is 350 $g \cdot m^{-2}$ for the optimal policy—about 10% below the maximal attainable yield. With irrigation costs of \$1020 ha^{-1} (about half of which is due to drainage), the net income (excluding labor and other inputs) from the optimal policy amounts to \$2480 ha^{-1}.

With the option to use saline water, we find that the optimal water mix is $s = 0.21$. The lower unit cost of water allows increasing slightly the turnpike moisture (to $\hat{\theta} = 0.15$) and the total amount of irrigation water so that the harvested yield increases marginally (to 351.3 $g \cdot m^{-2}$). However, this yield is obtained with a smaller irrigation bill of \$934 ha^{-1}, leaving a net income of \$2579 ha^{-1} to the growers.

The saving on freshwater may be even more important. Under the optimal mix, the saving on this precious resource amounts to 16%. Indeed, in arid regions such as the Arava valley, freshwater scarcity might dominate its nominal cost in determining the total quantity of applied irrigation. Scarcity turns into a binding constraint when growers are allocated an exogenous quota of freshwater below their use under the optimal (nonbinding) policy. The effect of the binding freshwater quota takes the form of a fixed shadow price to be added to the relative cost of water. Shani, Tsur and Zemel (2004) show that the shadow price should be adjusted so that the effective unit cost implies irrigation using exactly the allocated freshwater quota. As an example, assume that the quota amounts to only 75% of freshwater used for irrigation under the mixed-water policy discussed above. The constraint corresponds to adding a shadow price of \$0.42 m^{-3} to the nominal freshwater cost $W_f = \$0.3\ m^{-3}$. The increased effective cost implies an increase in salinity to $s = 0.282$ and an 18% decrease in the total amount of irrigation water, reducing the yield by 9% (to 318.6 $g \cdot m^{-2}$). Accounting for the smaller water bill, however, we find that the net income loss is a mere 5% (to \$2458 ha^{-1}).

It is instructive to compare this quota-bound policy with the outcome of the unbound policy based on freshwater only. Although the net incomes differ by less than 1%, the bound, mixed-water policy uses only 63% of freshwater required by the unbound policy. Indeed, if the same freshwater quota were imposed on growers without access to the saline resource, the net income would drop to \$1998 ha^{-1}, representing an income loss of nearly 20%. We see, therefore, that by carefully adjusting the turnpike policy, the growers can exploit the saline water resource to mitigate the significant losses implied by freshwater scarcity.

Finally, we remark that the same methodology can incorporate other effects of salinity. Assume, for example, that saline irrigation water is eventually drained to an underlying freshwater aquifer, or that it increases soil salinity for the following seasons, and the environmental damage is proportional to the cumulative amount of salt applied. The damage in such cases can be modeled as an additional fixed component of the unit cost of saline water. Except for changes in the numerical values of the optimal parameters, the characteristic turnpike policy and the solution methodology will not be affected.

5. Concluding comments

In a recent publication, Shani, Tsur and Zemel (2004) used Optimal Control theory to derive dynamic irrigation schemes that account for soil moisture and biomass growth dynamics as well as for the associated cost of irrigation water. The results reveal two important features: (i) Although the biomass and soil moisture dynamics are quite complex, the optimal policy displays an extremely simple turnpike behavior, and (ii) The turnpike policy is robust to a wide range of variations and extensions which, in spite of adding significant new considerations to the optimization tradeoffs, can modify the numerical values of the optimal parameters but not the characteristic behavior of the optimal policy.

The analysis applied here, based on Green's Theorem, provides a simple and elegant explanation to both features. The turnpike behavior follows from simple geometric considerations with little recourse to the complexities of the dynamic system. Evidently, the original method requires some modifications to cope with the new properties of the optimization problem considered here. In particular, we exploit the observation that turnpike trajectories not only provide the maximum for the objective, but also give rise to the minimal and maximal dynamically feasible process durations, as well as to the continuum of durations between these extremes. In fact, this observation follows from the same geometric considerations used to compare the objectives of competing trajectories.

We also note the simple relation between the Hamiltonian and the optimal turnpike state. The Hamiltonian function, which is the key element of Optimal Control theory, is typically absent in Green's Theorem analysis. Its role in the present formulation, thus, provides an interesting link between these two complementary methodologies of deriving optimal dynamic solutions.

It has been suggested (Haynes (1966), Sethi (1976)) that simple turnpike behavior can characterize the solutions of a variety of complex multi-dimensional dynamic optimization problems. Indeed, recent economic studies of optimal R&D strategies in the context of resource scarcity and economic growth (see Tsur and Zemel (2000), Tsur and Zemel (2002), Tsur and Zemel (2003), Tsur and Zemel (2004)) reveal interesting examples of such behavior for multi-dimensional infinite horizon problems with discounted utilities. Analyzing such problems in terms of Green or Stokes' Theorems (as in Haynes (1966) or Sethi (1976)) remains a challenge for future research.

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